End-Semester Exam | ISI Bangalore | 2nd May 2012

Representation Theory of finite groups | 100 Points

Notes.
(a) Begin each answer on a separate sheet.
(b) Justify all your steps. Assume only those results that have been proved in class. All other steps should be justified.
(c) $\mathbb{Z}$ = integers, $\mathbb{Q}$ = rational numbers, $\mathbb{R}$ = real numbers, $\mathbb{C}$ = complex numbers.
(d) Unless specifically mentioned, any group under consideration may be assumed to be finite and for representations, the underlying field may be assumed to be $\mathbb{C}$.
(e) The total number of points available to score is 150. You will be assigned

$$\text{Minimum } \{100, \text{ points scored by you}\}.$$

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1. [8 points] Prove that for any $n > 0$, $\mathbb{Q} \otimes_{\mathbb{Z}} (\mathbb{Z}/n\mathbb{Z}) \cong 0$.

2. [20 points] Let V, W be irreducible G-modules that are <u>not</u> isomorphic to each other. Prove that the only G-submodules of $M := V \oplus W$, other than 0 and M itself, are $V \cong V \oplus 0$ and $W \cong 0 \oplus W$.

3. [20 points]
   (i) Write the character table for S_3.
   (ii) Let V denote the standard module for S_3. Write the character of $V^{\otimes 10}$ and use it to identify the multiplicity of each of its irreducible components.

4. [8 points] Let k be an algebraically closed field of characteristic 0. Prove that $\mathrm{M}_2(k)$, the k-algebra of 2×2 matrices over k, is not isomorphic to the group ring of any finite group G over k.

5. [8 points] Prove or disprove: $V^{\otimes 3} \cong \mathrm{Sym}^3 V \oplus \wedge^3 V$ for every finite-dimensional vector space V over k.

6. [18 points] Let $G = C_2 \times C_2$ be the Klein 4-group where C_2 is the cyclic group of order 2. Describe all its irreducible representations over $\mathbb{C}$. (This means, list them up to isomorphism; for every irreducible representation V, give its dimension and give the action of each $g \in G$ on a suitable basis of V.)

7. [8 points] Let G be a finite group such that every complex irreducible representation of G has dimension 1. Prove that G is abelian.

8. [20 points] Let G be a finite group having an abelian subgroup N of index 2. Prove that every irreducible G-module over $\mathbb{C}$ has dimension at most 2.

9. [12 points] Let τ be the $\mathbb{C}$-algebra automorphism of polynomial rings $\mathbb{C}[X,Y] \to \mathbb{C}[X,Y]$ induced by the rule

$$\tau(X) = Y, \qquad\qquad \tau(Y) = X.$$

Then the subspace $V_d \subset \mathbb{C}[X,Y]$ of all polynomials of degree at most d becomes a representation for the cyclic group $C_2 = \{1, \tau\}$ via the above rule. Decompose V_2 as a direct sum of irreducible C_2-modules.

10. [16 points] Let G be a finite group.

(i) Let r be an element in the centre of $\mathbb{C}[G]$. Verify that for any G-module W, the map $\mu_r \colon W \to W$, given by multiplication by r, is G-linear.

(ii) If W in part (i) is irreducible, prove that $\mu_r = \lambda I_W$ for some $\lambda \in \mathbb{C}$ where I_W is the identity map on W.

11. [12 points] Let $\mathbb{L}_{-1}$ denote the 1-dimensional sign-representation of the symmetric group S_n and V the standard $(n-1)$-dimensional module for S_n. Prove that V and $V \otimes \mathbb{L}_{-1}$ are not isomorphic if $n > 3$.